Section 7.2
Areas in the Plane

Area between Curves

If f and g are continuous with $f(x) \geq g(x)$ throughout [a,b], then the area between the curves y = f(x) and y = g(x) from a to b is the integral of [f - g] from a to b,

$$\int = \int_a^b [f(x) - g(x)]\, dx$$

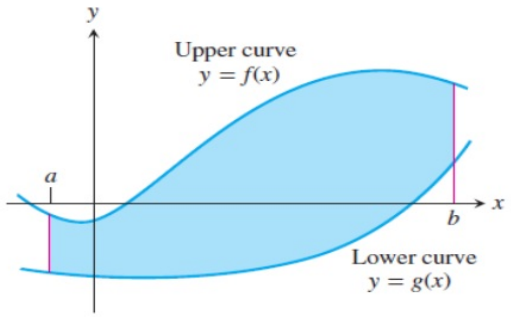


Figure 7.3 The region between $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

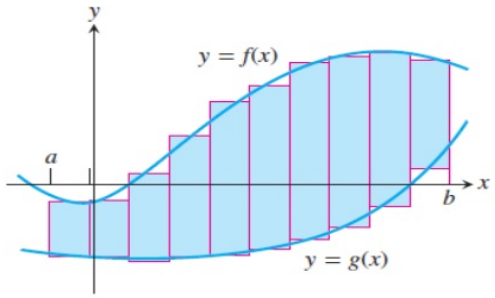


Figure 7.4 We approximate the region with rectangles perpendicular to the x-axis.

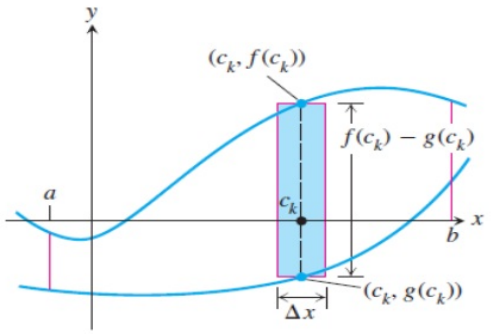
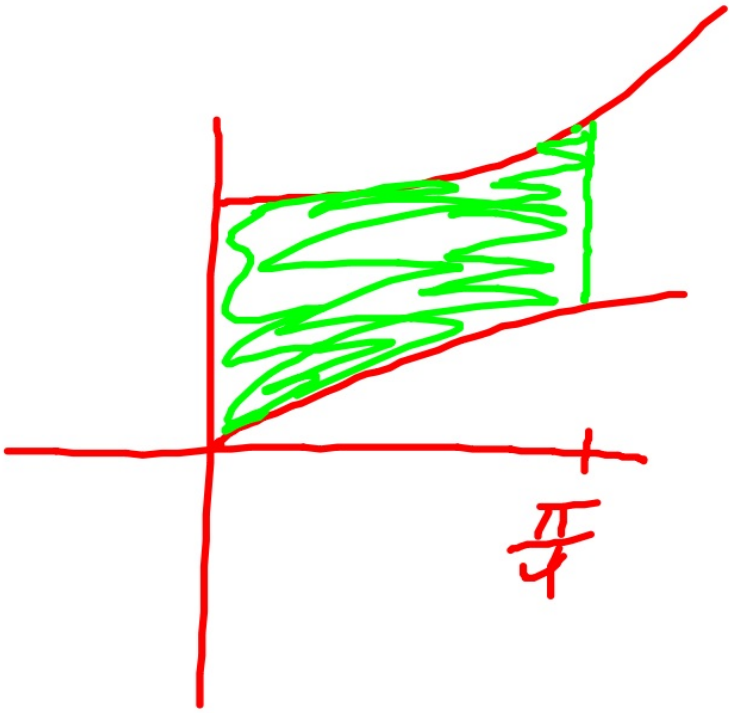


Figure 7.5 The area of a typical rectangle is $[f(c_k) - g(c_k)]\,\Delta x$.

Find the area of the region between $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \pi/4$.



$$\int_0^{\frac{\pi}{4}} (\sec^2 x - \sin x)\, dx$$

$$= \tan x + \cos x \Big]_0^{\frac{\pi}{4}}$$

$$= \left(\tan\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) - \left(\tan 0 + \cos 0\right)$$

Find the area of the region enclosed by intersecting curves, the intersection points give the limits of integration.

Find the area of the region enclosed by the graphs of $y = 2\cos x$ and $y = x^2 - 1$.

$$\int_{-1.265}^{1.265} (2\cos x - (x^2 - 1))\,dx$$

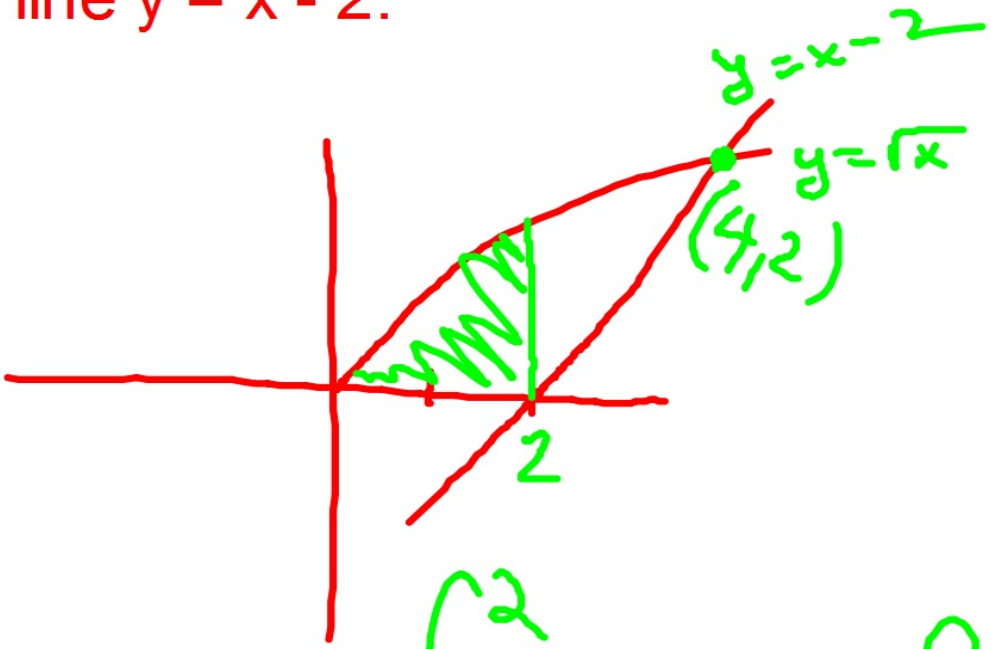
$$2\cos x - x^2 + 1$$

$$= 2\sin x - \frac{x^3}{3} + x \Big]_{-1.265}^{1.265}$$

$$= \left(2\sin 1.265 - \frac{1.265^3}{3} + 1.265\right) - \left(2\sin(-1.265) - \frac{(-1.265)^3}{3} - 1.\right.$$

If a boundary of a region is defined by more than one function, we can partition the region into subregions that correspond to the function changes and proceed as usual

Find the area of the region R in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

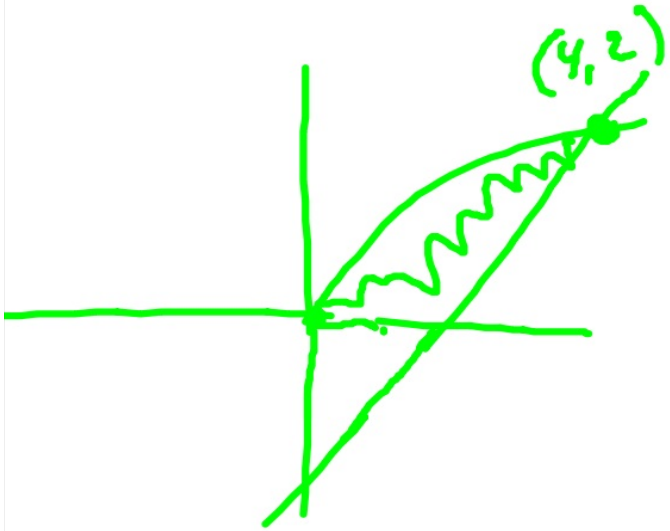


$$\int_0^2 \sqrt{x}\,dx + \int_2^4 \left(\sqrt{x} - (x-2)\right)dx$$

Find the area of the region in the previous example by integrating with respect to y.

$y = \sqrt{x} \longrightarrow x = y^2$

$y = x - 2 \longrightarrow x = y + 2$



$$\int_0^2 \left((y+2) - (y^2) \right) dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big]_0^2$$

Find the area between $y = \sqrt{x + 4}$ and $y = x^2$

$$\int_{-1.284}^{1.534} \left(\left(\sqrt{x+4} \right) - \left(x^2 \right) \right) dx$$

$$= \left[\frac{2(x+4)^{\frac{3}{2}}}{3} - \frac{x^3}{3} \right]$$

Find the area enclosed by $y = 2 - x^2$ and $y = -x$.

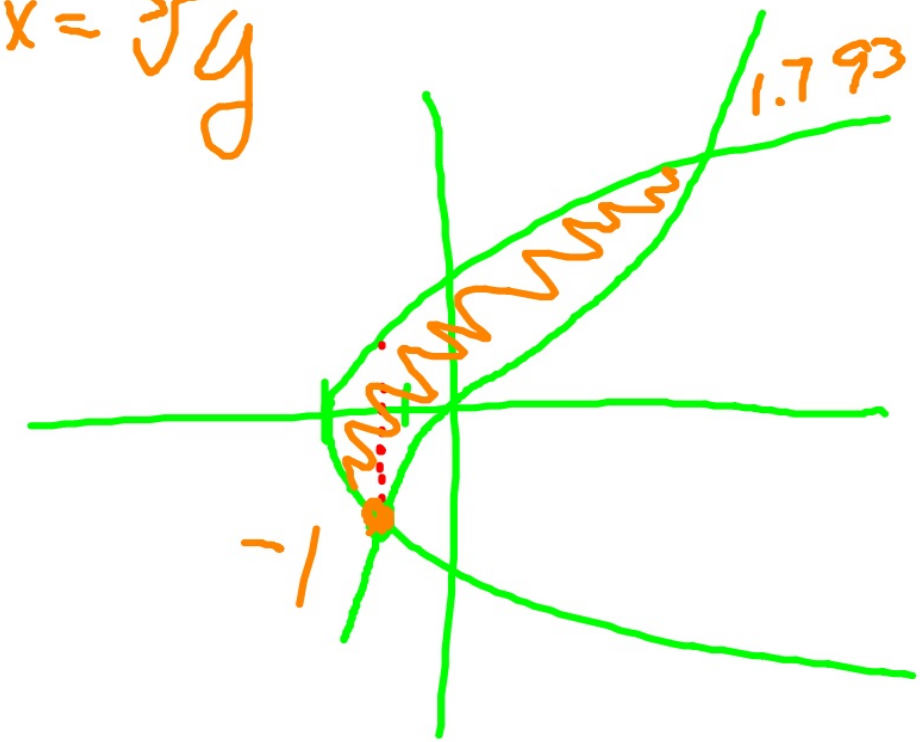
Integrating with Respect to y

Find the area $x = y^2$ and $y = x - 2$ and $y = 0$

Find the area of

$y = x^3$ and $y = \sqrt{x + 2}$ and $y = -\sqrt{x + 2}$

$x = \sqrt[3]{y}$



$x = y^2 - 2$

$$\int_{-1}^{1.793} \left(\sqrt[3]{y} - (y^2 - 2)\right) dy$$

Find the area of the region enclosed by the graphs of $y = x^3$ and $x = y^2 - 2$.